A STUDY OF THE SCATTERING OF SOUND WAVES IN FLUIDS PART I - SCATTERING OF A PLANE WAVE BY A SPHERICAL BUBBLE

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ABSTRACT

The scattering of a plane wave by a spherical bubble immersed in a homogeneous perfect fluid of infinite extent has been examined. Expressions for (1) the average sound intensity of the scattered wave, and (2) the average total intensity of the entire sound field have been obtained. Numerical calculations have been carried out for the case of long wavelengths and for distances close to the bubble. Several curves have been plotted.

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PART I - SCATTERING OF A PLANE WAVE BY A SPHERICAL BUBBLE

TABLE OF CONTENTS

A STUDY OF THE SCATTERING OF SOUND WAVES IN FLUIDS
PART I - SCATTERING OF A PLANE WAVE BY A SPHERICAL BUBBLE

I. INTRODUCTION

The scattering of a plane wave by a spherical bubble immersed in a homogeneous perfect fluid of infinite extent has been examined. An expression for the average sound intensity of the scattered wave very close to the bubble has been obtained. The important case of long wavelengths or small bubble radii is of particular interest and has been given special attention. Numerical results are presented in the form of polar diagrams for several different values of r/a and $2\pi a/\lambda$, where r is the distance from the center of the bubble, a is the bubble radius and λ is the wavelength.

In addition to the average intensity of the scattered wave, an expression for the average total intensity of the entire sound field has been obtained. This result is compared with the average sound intensity of a plane wave travelling in an infinite homogeneous fluid with no bubble present. It is found that the influence of the bubble on the average total sound intensity is negligible when $2\pi a/\lambda$ is small and $r/a \gg 10$. Thus, the average total sound intensity differs very little from that due to a plane wave travelling in an infinite homogeneous fluid with no bubble present. For small values of r/a, however, the presence of the bubble seriously alters the magnitude of the average total sound intensity from that due to a plane wave. Polar diagrams of the ratio of the average total sound intensity to the average intensity of the incident plane wave are presented for several different values of r/a.

II. DISCUSSION

Consider a plane wave incident upon a spherical bubble of radius a immersed in an infinite homogeneous perfect fluid characterized by density ρ and velocity c. It is desired to find expressions for the average sound intensity of the scattered wave and the average total intensity of the entire sound field very close to the bubble. The special case when the ratio of wavelength to sphere radius is large is to be given special attention. Large values of this ratio imply that either the wavelength is long or the sphere radius is small.

The pressure equation for a sound wave is given by

$$\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} , \tag{1}$$

where p is the excess pressure and c is the velocity of sound. If the pressure dependence on time is of the form $\exp(-i\omega t)$, then

$$\nabla^2 p + k^2 p = 0 , \tag{2}$$

where

$$k = \frac{\omega}{c} = \frac{2\pi\nu}{c} = \frac{2\pi}{\lambda} . \tag{3}$$

The scalar wave equation in spherical coordinates is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial p}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 p}{\partial\phi^2} + k^2 p = 0 , \tag{4}$$

where r, θ and ϕ are shown in Figure 1. The problem under discussion possesses symmetry about the polar axis so the incident and scattered waves are independent of ϕ. In order to solve the problem it is necessary to find solutions of the scalar wave equation that represent an incoming plane wave and an outgoing scattered wave. In addition to those requirements, it is necessary that the solutions satisfy the proper boundary conditions on the surface of the bubble.

The expression for a plane wave travelling to the right along the polar axis in terms of spherical waves is given by

$$p_p = P_o \, e^{ik(r\cos\theta - ct)}$$

or (5)

$$p_p = P_o \sum_{m=0}^{\infty} (2m+1) i^m P_m(\cos\theta) j_m(kr) e^{-i\omega t},$$

where $P_m(\cos\theta)$ are Legendre Functions of the First Kind and $j_m(kr)$ are Spherical Bessel Functions. The expression for the wave scattered by a sphere of radius a whose center is at the polar origin is

$$p_s = \sum_{m=0}^{\infty} C_m P_m(\cos\theta)\left[j_m(kr) + i n_m(kr)\right] e^{-i\omega t} \tag{6}$$

where $n_m(kr)$ are Spherical Neumann Functions. The combination $[j_m(kr) + i n_m(kr)]$ represents an outgoing wave.

The appropriate boundary condition for the bubble is that the total pressure on its surface be zero. Therefore, the condition to be satisfied when $r = a$ is that the sum of the incident and scattered pressures be zero.

$$p_p + p_s = 0 \qquad , r = a \quad . \tag{7}$$

The constants C_m can be found by substituting equations (5) and (6) into equation (7). Substitution yields

$$P_0 \sum_{m=0}^{\infty} (2m+1) i^m P_m(\cos\theta) j_m(ka) e^{-i\omega t}$$

$$+ \sum_{m=0}^{\infty} C_m P_m(\cos\theta) \left[j_m(ka) + i n_m(ka) \right] e^{-i\omega t} = 0$$

or

$$P_0 (2m+1) i^m j_m(ka) + C_m \left[j_m(ka) + i n_m(ka) \right] = 0. \tag{8}$$

Let

$$\left[j_m(kr) + i n_m(kr) \right] = - i A_m(kr) e^{i\alpha_m(kr)} , \tag{9}$$

where

$$A_m(kr) = \sqrt{\left[j_m(kr) \right]^2 + \left[n_m(kr) \right]^2} \tag{10}$$

$$\alpha_m(kr) = \tan^{-1} \frac{j_m(kr)}{-n_m(kr)} \quad . \tag{11}$$

By making use of the relations

$$j_m(kr) = A_m(kr) \sin \alpha_m(kr) \tag{12}$$

$$i^{m+1} = e^{i\frac{\pi}{2}(m+1)} \quad , \tag{13}$$

the constants C_m can be written

$$C_m = -P_o(2m+1)e^{-i\left[\alpha_m(ka)-\frac{\pi}{2}(m+1)\right]}\sin\alpha_m(ka)\ . \tag{14}$$

The expression for the scattered pressure wave becomes

$$p_s = -P_o\sum_{m=0}^{\infty}(2m+1)e^{-i\left[\alpha_m(ka)-\alpha_m(kr)-\frac{\pi}{2}m\right]}$$
$$\cdot A_m(kr)\sin\alpha_m(ka)\,P_m(\cos\theta)\,e^{-i\omega t}\ . \tag{15}$$

The radial component of velocity for the scattered wave is given by

$$u_{s_r} = \frac{1}{i\omega\rho}\frac{\partial p_s}{\partial r}\ . \tag{16}$$

By substituting equation (6) into (16) the expression for the velocity of the scattered wave becomes

$$u_{s_r} = \frac{k}{i\omega\rho}\sum_{m=0}^{\infty}C_m P_m(\cos\theta)\frac{d}{d(kr)}\left[j_m(kr)+in_m(kr)\right]e^{-i\omega t}$$

or

$$u_{s_r} = \frac{1}{i\rho c}\sum_{p=0}^{\infty}C_p P_p(\cos\theta)\left[j_p'(kr)+in_p'(kr)\right]e^{-i\omega t}\ . \tag{17}$$

It is convenient to introduce the following notation:

$$(-i)\left[j_p'(kr)+in_p'(kr)\right] = B_p(kr)\,e^{i\beta_p(kr)}\ . \tag{18}$$

By using the appropriate recurrence formulas it can be shown that

$$B_p(kr) = \frac{1}{(2p+1)}\sqrt{\left[p n_{p-1}(kr) - (p+1) n_{p+1}(kr)\right]^2 + \left[(p+1) j_{p+1}(kr) - p j_{p-1}(kr)\right]^2} \quad \ldots\ldots \quad (19)$$

and

$$\beta_p(kr) = \tan^{-1} \frac{(p+1) j_{p+1}(kr) - p j_{p-1}(kr)}{p n_{p-1}(kr) - (p+1) n_{p+1}(kr)} . \quad (20)$$

Substituting (14) and (18) into (17), the expression for the radial velocity of the scattered wave becomes

$$u_{s_r} = -\frac{P_0}{\rho c} \sum_{p=0}^{\infty} (2p+1) e^{-i\left[\alpha_p(ka) - \beta_p(kr) - \frac{\pi}{2}(p+1)\right]}$$

$$\cdot B_p(kr) \sin \alpha_p(ka) P_p(\cos\theta) e^{-i\omega t} . \quad (21)$$

In a similar way it is found that the radial component of velocity for the incident plane wave is

$$u_{p_r} = -\frac{P_0}{\rho c} \sum_{p=0}^{\infty} (2p+1) P_p(\cos\theta) j_p^{\nabla}(kr) e^{i\frac{\pi}{2}(p+1)} e^{-i\omega t} \quad (22)$$

where

$$j_p^{\nabla}(kr) = \frac{1}{(2p+1)}\left[p j_{p-1}(kr) - (p+1) j_{p+1}(kr)\right] . \quad (23)$$

The angular component of velocity for the scattered wave is given by

$$u_{s_\theta} = \frac{1}{i\omega\rho}\,\frac{1}{r}\,\frac{\partial p_s}{\partial \theta} \quad . \tag{24}$$

Differentiating equation (15) with respect to θ and combining terms

$$u_{s_\theta} = +\frac{P_0}{\rho c}\,\frac{1}{kr}\sum_{p=0}^{\infty}(2p+1)\,e^{-i\left[\alpha_p(ka)-\alpha_p(kr)-\frac{\pi}{2}(p+1)\right]}$$

$$\cdot A_p(kr)\sin\alpha_p(ka)\,P_p'(\cos\theta)\,e^{-i\omega t} \tag{25}$$

where

$$P_p'(\cos\theta) = \frac{\partial}{\partial\theta}P_p(\cos\theta) \,.$$

The angular component of velocity for the incident plane wave is

$$u_{p_\theta} = -\frac{P_0}{\rho c}\,\frac{1}{kr}\sum_{p=0}^{\infty}(2p+1)P_p'(\cos\theta)\,j_p(kr)\,e^{i\frac{\pi}{2}(p+1)}\,e^{-i\omega t} \,. \tag{26}$$

The average intensity of the scattered wave can be found by multiplying the real parts of the expressions for the scattered pressure and scattered radial velocity and averaging over time. Note that the average scattered intensity is the average rate at which energy is transmitted per square centimeter in a radial outward direction. The real parts of the expressions for p_s and u_s are given by

$$Re(p_s) = -P_0 \sum_{m=0}^{\infty} (2m+1) \sin \alpha_m(ka) P_m(\cos\theta) A_m(kr)$$

$$\cdot \cos\left[\alpha_m(ka) - \alpha_m(kr) - \frac{\pi}{2}m + \omega t\right] \tag{27}$$

and

$$Re(u_s) = -\frac{P_0}{\rho c} \sum_{p=0}^{\infty} (2p+1) \sin \alpha_p(ka) P_p(\cos\theta) B_p(kr) \tag{28}$$

$$\cdot \cos\left[\alpha_p(ka) - \beta_p(kr) - \frac{\pi}{2}(p+1) + \omega t\right] .$$

Therefore, the intensity of the scattered wave becomes

$$Re(p_s)\, Re(u_s) = \frac{P_0^2}{\rho c} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} (2m+1)(2p+1) \sin\alpha_m(ka) \sin\alpha_p(ka)$$

$$P_m(\cos\theta) P_p(\cos\theta) A_m(kr) B_p(kr) \cos\left[\alpha_m(ka) - \alpha_m(kr) - \frac{\pi}{2}m + \omega t\right]$$

$$\cdot \cos\left[\alpha_p(ka) - \beta_p(kr) - \frac{\pi}{2}(p+1) + \omega t\right] . \tag{29}$$

By using the appropriate trigonometric formulae and omitting all terms whose time average is zero,

$$\cos\left[\alpha_m(ka) - \alpha_m(kr) - \frac{\pi}{2}m + \omega t\right] \cos\left[\alpha_p(ka) - \beta_p(kr) - \frac{\pi}{2}(p+1) + \omega t\right]$$

$$= \frac{1}{2} \cos\left[\alpha_m(ka) - \alpha_p(ka) - \alpha_m(kr) + \beta_p(kr) - \frac{\pi}{2}m + \frac{\pi}{2}(p+1)\right] . \tag{30}$$

Substituting the above result into equation (29) it is found that the average intensity of the scattered wave is

$$I_r = I_0 \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} (2m+1)(2p+1) \sin \alpha_m(ka) \sin \alpha_p(ka)$$

$$\cdot P_m(\cos\theta) P_p(\cos\theta) A_m(kr) B_p(kr)$$

$$\cdot \cos\left[\alpha_m(ka) - \alpha_p(ka) - \alpha_m(kr) + \beta_p(kr) - \frac{\pi}{2} + \frac{\pi}{2}(p+1)\right], \tag{31}$$

where $I_0 = P_0^2/2\rho c$ is the average intensity of the incident plane wave. Note that if c.g.s. units are used, the intensity is measured in ergs per second per square centimeter.

The average total intensity of the entire sound field can be found by taking the vector sum of the average total radial intensity and the average total angular intensity. The average total radial intensity can be found by multiplying the real parts of the expressions for the total pressure and total radial velocity and averaging over time. Note that a positive value of the average total radial intensity is the average rate at which energy is transmitted per unit area in a radial outward direction. The average total angular intensity can be found by multiplying the real parts of the expressions for the total pressure and total angular velocity and averaging over time. In this case a positive value of the total angular intensity is the average rate at which energy is transmitted per unit area in the direction of increasing θ.

The total pressure, total radial velocity and total angular velocity are given by the following expressions:

$$
\begin{aligned}
p_t &= p_i + p_s \\
&= P_0 \sum_{m=0}^{\infty} (2m+1) P_m(\cos\theta)\, j_m(kr)\, e^{i\frac{\pi}{2}m}\, e^{-i\omega t} \\
&\quad - P_0 \sum_{m=0}^{\infty} (2m+1) P_m(\cos\theta)\, A_m(kr) \sin\alpha_m(ka) \\
&\quad \cdot e^{-i\left[\alpha_m(ka) - \alpha_m(kr) - \frac{\pi}{2}m\right]}\, e^{-i\omega t}
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
u_{t_r} &= u_{i_r} + u_{s_r} \\
&= -\frac{P_0}{\rho c} \sum_{p=0}^{\infty} (2p+1) P_p(\cos\theta)\, j_p'(kr)\, e^{i\frac{\pi}{2}(p+1)}\, e^{-i\omega t} \\
&\quad - \frac{P_0}{\rho c} \sum_{p=0}^{\infty} (2p+1) P_p(\cos\theta)\, B_p(kr) \sin\alpha_p(ka) \\
&\quad \cdot e^{-i\left[\alpha_p(ka) - \beta_p(kr) - \frac{\pi}{2}(p+1)\right]}\, e^{-i\omega t}.
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
u_{t_\theta} &= u_{i_\theta} + u_{s_\theta} \\
&= -\frac{P_0}{\rho c} \frac{1}{kr} \sum_{p=0}^{\infty} (2p+1) P_p'(\cos\theta)\, j_p(kr)\, e^{i\frac{\pi}{2}(p+1)}\, e^{-i\omega t} \\
&\quad + \frac{P_0}{\rho c} \frac{1}{kr} \sum_{p=0}^{\infty} (2p+1) P_p'(\cos\theta)\, A_p(kr) \sin\alpha_p(ka) \\
&\quad \cdot e^{-i\left[\alpha_p(ka) - \alpha_p(kr) - \frac{\pi}{2}(p+1)\right]}\, e^{-i\omega t}.
\end{aligned} \tag{34}
$$

The real parts of these quantities are given by

$$\mathrm{Re}(p_t) = P_0 \sum_{m=0}^{\infty} (2m+1) P_m(\cos\theta) j_m(kr) \cos\left[-\tfrac{\pi}{2}m + \omega t\right]$$

$$- P_0 \sum_{m=0}^{\infty} (2m+1) P_m(\cos\theta) A_m(kr) \sin\alpha_m(ka)$$

$$\cdot \cos\left[\alpha_m(ka) - \alpha_m(kr) - \tfrac{\pi}{2}m + \omega t\right] \tag{35}$$

$$\mathrm{Re}(u_{t_r}) = -\frac{P_0}{\rho c} \sum_{p=0}^{\infty} (2p+1) P_p(\cos\theta) j_p'(kr) \cos\left[-\tfrac{\pi}{2}(p+1) + \omega t\right]$$

$$- \frac{P_0}{\rho c} \sum_{p=0}^{\infty} (2p+1) P_p(\cos\theta) B_p(kr) \sin\alpha_p(ka)$$

$$\cdot \cos\left[\alpha_p(ka) - \beta_p(kr) - \tfrac{\pi}{2}(p+1) + \omega t\right] \tag{36}$$

$$\mathrm{Re}(u_{t_\theta}) = -\frac{P_0}{\rho c}\frac{1}{kr} \sum_{p=0}^{\infty} (2p+1) P_p'(\cos\theta) j_p(kr) \cos\left[-\tfrac{\pi}{2}(p+1) + \omega t\right]$$

$$+ \frac{P_0}{\rho c}\frac{1}{kr} \sum_{p=0}^{\infty} (2p+1) P_p'(\cos\theta) A_p(kr) \sin\alpha_p(ka)$$

$$\cdot \cos\left[\alpha_p(ka) - \alpha_p(kr) - \tfrac{\pi}{2}(p+1) + \omega t\right] . \tag{37}$$

The total radial intensity becomes

$$
\begin{aligned}
&\mathcal{R}e(p_t)\,\mathcal{R}e(u_{tr}) \\
&= -\frac{P_0}{\rho c}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(2m+1)(2p+1)P_m(\cos\theta)P_p(\cos\theta)j_m(kr)j_p'(kr) \\
&\quad\cdot\cos\left[-\frac{\pi}{2}m+\omega t\right]\cos\left[-\frac{\pi}{2}(p+1)+\omega t\right] \\
&+\frac{P_0^2}{\rho c}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(2m+1)(2p+1)P_m(\cos\theta)P_p(\cos\theta)A_m(kr)B_p(kr) \\
&\quad\cdot\sin\alpha_m(ka)\sin\alpha_p(ka)\cos\left[\alpha_m(ka)-\alpha_m(kr)-\frac{\pi}{2}m+\omega t\right] \\
&\quad\cdot\cos\left[\alpha_p(ka)-\beta_p(kr)-\frac{\pi}{2}(p+1)+\omega t\right] \\
&-\frac{P_0^2}{\rho c}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(2m+1)(2p+1)P_m(\cos\theta)P_p(\cos\theta)j_m(kr)B_p(kr)\sin\alpha_p(ka) \\
&\quad\cdot\cos\left[-\frac{\pi}{2}m+\omega t\right]\cos\left[\alpha_p(ka)-\beta_p(kr)-\frac{\pi}{2}(p+1)+\omega t\right] \\
&+\frac{P_0^2}{\rho c}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(2m+1)(2p+1)P_m(\cos\theta)P_p(\cos\theta)A_m(kr)\sin\alpha_m(ka)j_p'(kr) \\
&\quad\cdot\cos\left[\alpha_m(ka)-\alpha_m(kr)-\frac{\pi}{2}m+\omega t\right]\cos\left[-\frac{\pi}{2}(p+1)+\omega t\right].
\end{aligned}
\qquad \ldots\ldots \quad (38)
$$

The total angular intensity becomes

$$
\begin{aligned}
&\mathcal{R}e\,(p_t)\,\mathcal{R}e\,(u_{t\theta}) \\
&= -\frac{P_0^2}{\rho c}\frac{1}{kr}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(2m+1)(2p+1)P_m(\cos\theta)P_p'(\cos\theta)j_m(kr)j_p(kr) \\
&\quad\cdot\cos\left[-\frac{\pi}{2}m+\omega t\right]\cos\left[-\frac{\pi}{2}(p+1)+\omega t\right] \\
&\quad-\frac{P_0^2}{\rho c}\frac{1}{kr}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(2m+1)(2p+1)P_m(\cos\theta)P_p'(\cos\theta)A_m(kr)A_p(kr) \\
&\quad\cdot\sin\alpha_m(ka)\sin\alpha_p(ka)\cos\left[\alpha_m(ka)-\alpha_m(kr)-\frac{\pi}{2}m+\omega t\right] \\
&\quad\cdot\cos\left[\alpha_p(ka)-\alpha_p(kr)-\frac{\pi}{2}(p+1)+\omega t\right] \\
&\quad+\frac{P_0^2}{\rho c}\frac{1}{kr}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(2m+1)(2p+1)P_m(\cos\theta)P_p'(\cos\theta)j_m(kr)A_p(kr)\sin\alpha_p(ka) \\
&\quad\cdot\cos\left[-\frac{\pi}{2}m+\omega t\right]\cos\left[\alpha_p(ka)-\alpha_p(kr)-\frac{\pi}{2}(p+1)+\omega t\right] \\
&\quad+\frac{P_0^2}{\rho c}\frac{1}{kr}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(2m+1)(2p+1)P_m(\cos\theta)P_p'(\cos\theta)A_m(kr)\sin\alpha_m(ka)j_p(kr) \\
&\quad\cdot\cos\left[\alpha_m(ka)-\alpha_m(kr)-\frac{\pi}{2}m+\omega t\right]\cos\left[-\frac{\pi}{2}(p+1)+\omega t\right].
\end{aligned}
$$

..... (39)

Note that in equations (38) and (39) the first double summation is the intensity of the incident wave and the second double summation is the intensity of the scattered wave. The last two double summations represent interaction between the incident and scattered fields. The presence of interaction terms in the expressions for the intensity components is due to the fact that the total pressure and component particle velocities each involve two terms. Upon multiplication, four products are obtained, two of which are cross product terms that represent interaction between the incident and scattered fields. The contribution of these terms to the total intensity field is significant when intentity measurements are made close to the bubble. It should be noted that the total pressure and component particle velocities do not involve cross product terms since these can be expressed as sums of the incident and scattered fields. In order to find the average total radial and angular intensities, the time averages of equations (38) and (39) must be determined.

The average total radial intensity is found to be

$$I_r = -\frac{P_o^2}{2\rho c}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(2m+1)(2p+1)P_m(\cos\theta)P_p(\cos\theta)j_m(kr)j_p'(kr)$$

$$\cdot\cos\left[-\frac{\pi}{2}m+\frac{\pi}{2}(p+1)\right]$$

$$+\frac{P_o^2}{2\rho c}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(2m+1)(2p+1)P_m(\cos\theta)P_p(\cos\theta)A_m(kr)B_p(kr)\sin\alpha_m(ka)$$

$$\cdot\sin\alpha_p(ka)\cos\left[\alpha_m(ka)-\alpha_m(kr)-\frac{\pi}{2}m-\alpha_p(ka)+\beta_p(kr)+\frac{\pi}{2}(p+1)\right]$$

(continued)

$$-\frac{P_o^2}{2\rho c}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(2m+1)(2p+1)P_m(\cos\theta)P_p(\cos\theta)j_m(kr)B_p(kr)$$

$$\cdot\sin\alpha_p(ka)\cos\left[-\frac{\pi}{2}m-\alpha_p(ka)+\beta_p(kr)+\frac{\pi}{2}(p+1)\right]$$

$$+\frac{P_o^2}{2\rho c}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(2m+1)(2p+1)P_m(\cos\theta)P_p(\cos\theta)A_m(kr)\sin\alpha_m(ka)$$

$$\cdot j_p'(kr)\cos\left[\alpha_m(ka)-\alpha_m(kr)-\frac{\pi}{2}m+\frac{\pi}{2}(p+1)\right].$$

$$\cdots\cdots \qquad (40)$$

The average total angular intensity is given by

$$I_\theta=-\frac{P_o^2}{2\rho c}\frac{1}{kr}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(2m+1)(2p+1)P_m(\cos\theta)P_p'(\cos\theta)j_m(kr)j_p(kr)$$

$$\cdot\cos\left[-\frac{\pi}{2}m+\frac{\pi}{2}(p+1)\right]$$

$$-\frac{P_o^2}{2\rho c}\frac{1}{kr}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(2m+1)(2p+1)P_m(\cos\theta)P_p'(\cos\theta)A_m(kr)A_p(kr)\sin\alpha_m(ka)$$

$$\cdot\sin\alpha_p(ka)\cos\left[\alpha_m(ka)-\alpha_m(kr)-\frac{\pi}{2}m-\alpha_p(ka)+\alpha_p(kr)+\frac{\pi}{2}(p+1)\right]$$

$$+\frac{P_o^2}{2\rho c}\frac{1}{kr}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(2m+1)(2p+1)P_m(\cos\theta)P_p'(\cos\theta)j_m(kr)A_p(kr)$$

$$\cdot\sin\alpha_p(ka)\cos\left[-\frac{\pi}{2}m-\alpha_p(ka)+\alpha_p(kr)+\frac{\pi}{2}(p+1)\right]$$

(continued)

$$+\frac{P_0^2}{2\rho c}\frac{1}{kr}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(2m+1)(2p+1)P_m(\cos\theta)P_p'(\cos\theta)A_m(kr)\sin\alpha_m(ka)$$

$$\cdot j_p(kr)\cos\left[\alpha_m(ka)-\alpha_m(kr)-\frac{\pi}{2}m+\frac{\pi}{2}(p+1)\right]$$

$$\ldots\ldots \qquad (41)$$

The quantity $P_0^2/2\rho c$ is the average intensity of the incident plane wave. If c.g.s. units are used, the intensity is measured in ergs per second per square centimeter.

The case of very long wavelengths is of particular interest. The expressions for the intensities can be simplified slightly by making use of well known asymptotic formulae. The formulae needed are the following:

$$\left.\begin{array}{llll} j_0(ka) \longrightarrow 1 & & m=0 & ka\to 0 \\ j_m(ka) \longrightarrow \dfrac{(ka)^m}{1\cdot 3\cdot 5\cdots(2m+1)} & & m>0 & ka\to 0 \\ n_0(ka) \longrightarrow -\dfrac{1}{ka} & & m=0 & ka\to 0 \\ n_m(ka) \longrightarrow -\dfrac{1\cdot 3\cdot 5\cdots(2m-1)}{(ka)^{m+1}} & & m>0 & ka\to 0 \end{array}\right\} \quad (42)$$

Substituting these results into equation (11)

$$\left.\begin{array}{ll} \tan\alpha_0(ka) \to (ka) & m=0 \\ \tan\alpha_m(ka) \to \dfrac{(ka)^{2m+1}}{[1\cdot 3\cdot 5\cdots(2m+1)][1\cdot 3\cdot 5\cdots(2m-1)]} & m>0 \end{array}\right\} \quad (43)$$

for small values of ka. Since the arguments of the tangents are small

$$\left.\begin{aligned} &\alpha_0(ka) \approx \sin\alpha_0(ka) \approx (ka) && m=0 \\ &\alpha_m(ka) \approx \sin\alpha_m(ka) \approx \frac{(ka)^{2m+1}}{[1\cdot3\cdot5\cdots(2m+1)][1\cdot3\cdot5\cdots(2m-1)]}, && \\ & && m>0 \end{aligned}\right\} \tag{44}$$

By making use of the above approximations, the terms in equations (35) and (36) involving $\alpha_m(ka)$ can be computed with ease for the case of long wavelengths.

The average total intensity of the entire sound field is given by

$$\vec{I}_T = \vec{i}_r I_r + \vec{i}_\theta I_\theta \quad , \tag{45}$$

where $\vec{i}_r$ and $\vec{i}_\theta$ are the unit vectors in the direction of increasing r and θ. The magnitude and phase of $\vec{I}_T$ are

$$|\vec{I}_T| = \sqrt{I_r^2 + I_\theta^2} \tag{46}$$

$$\Phi = \tan^{-1}\frac{I_\theta}{I_r} \quad . \tag{47}$$

In the above equations $|\vec{I}_T|$ is the magnitude of the average total intensity at the point (r,θ) and Φ is the angle between the average total intensity and the unit vector $\vec{i}_r$. The angular deviation between the direction of the average total intensity produced by a plane wave incident upon a spherical bubble and the average intensity of the same plane wave with no bubble present is given by

$$\Delta = \theta + \Phi \quad . \qquad (48)$$

The quantity Δ will be called the Intensity Deviation Angle.

III. RESULTS

The ratio of average scattered intensity to average incident intensity has been plotted as a function of angle for different values of $2\pi a/\lambda$ and r/a. Two sets of curves have been drawn and are presented in Figures 2 and 3. In the first set of curves $2\pi a/\lambda$ is held constant and r/a is varied; in the second set of curves r/a is held constant and $2\pi a/\lambda$ is varied. The following sets of values have been taken for purposes of calculation:

Figure 2: $2\pi a/\lambda = 0.10$, $r/a = 3/2, 2, 4$

Figure 3: $r/a = 2.00$, $2\pi a/\lambda = 0.05, 0.01, 0.02$

For the case in which r/a is fixed, there seems to be little change in the ratio of scattered intensities for the values of $2\pi a/\lambda$ considered. Note that for a wavelength of 100 feet and a bubble diameter of 3 feet, the value of $2\pi a/\lambda$ is approximately 0.10. For this value the ratio of intensities 1.5 feet away from the surface of the bubble in the backward direction along the polar axis is 0.309.

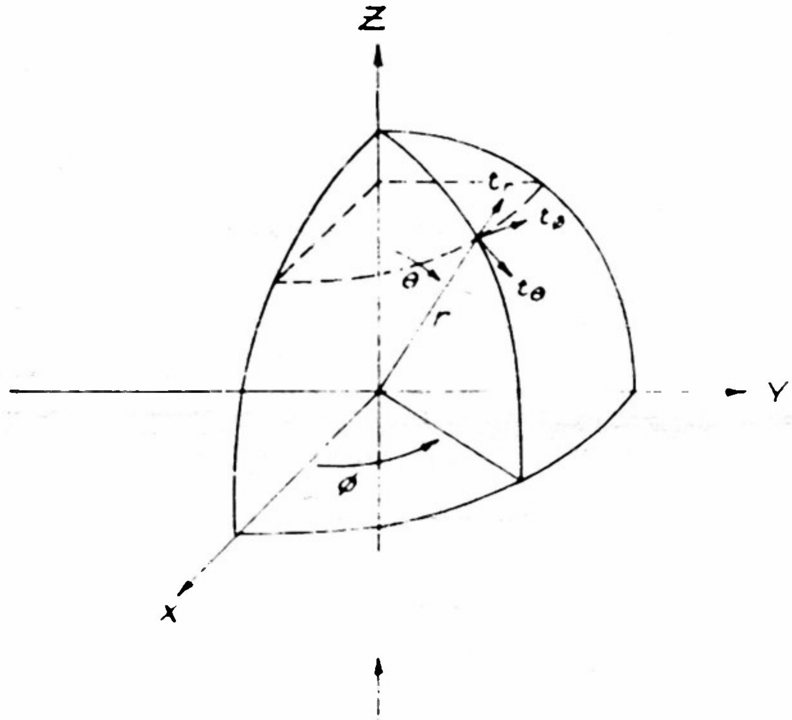
The average total sound intensity of the entire sound field is equal to the vector sum of the average total radial and average total angular intensity components. Unfortunately, the numerical calculation of these intensity components is lengthy and tedious since each component is equal to the sum of four double summations in which the indices run from zero to infinity. For computational purposes, however, only a finite number of these terms need to be taken because terms involving large values of the indices are extremely small. It is found that for small values of the parameters $2\pi a/\lambda$ and r/a, sufficient accuracy is obtained by taking the largest value of each index to be two. Each intensity component can, therefore, be expressed as the sum of thirty-six terms provided calculations are limited to the case of long wavelengths and distances close to the bubble. As the value of either $2\pi a/\lambda$ or r/a is increased, it is found that more and more terms are needed to obtain the correct values of the intensity components. Numerical work becomes prohibitive when the wavelength is of the same order of magnitude as the bubble radius.

In order to obtain some idea as to the influence of a spherical bubble on the average total intensity, calculations have been performed for values of $2\pi a/\lambda = 0.1$ and $r/a = 2, 4, 10$. These values are representative of a small bubble in a sound field of long wavelength and correspond approximately to a bubble diameter of 3 feet and a wavelength of 100 feet. Figure 4 is a plot of the Intensity Deviation Angle as a function of the polar angle θ for $2\pi a/\lambda = 0.1$ and $r/a = 2$. Figure 5 is a polar plot of the ratio of the average total sound intensity to the average intensity of the incident plane wave for $2\pi a/\lambda = 0.1$ and $r/a = 2$. Note that the arrows shown on the polar diagram indicate the direction of the total sound intensity vector with respect to the polar axis at the point $(2a, \theta)$. The value of $|\vec{I}_T|/I_0$ 1.5 feet away from the surface of the bubble in the backward direction along the polar axis is 0.185. Figure 6 is a polar plot of the ratio of the average total sound intensity to the average intensity of the incident plane wave for $2\pi a/\lambda = 0.1$ and $r/a = 2, 4, 10, \infty$. It is evident that for large values of r/a the presence of the bubble has little influence on the average total sound intensity.

IV. APPENDIX

APPENDIX

TABLE OF CONTENTS



DIRECTION OF
PLANE WAVE

FIGURE 1

RATIO OF AVERAGE SCATTERED INTENSITY TO AVERAGE INCIDENT INTENSITY AS A FUNCTION OF ANGLE ($2\pi a/\lambda = 0.10$)

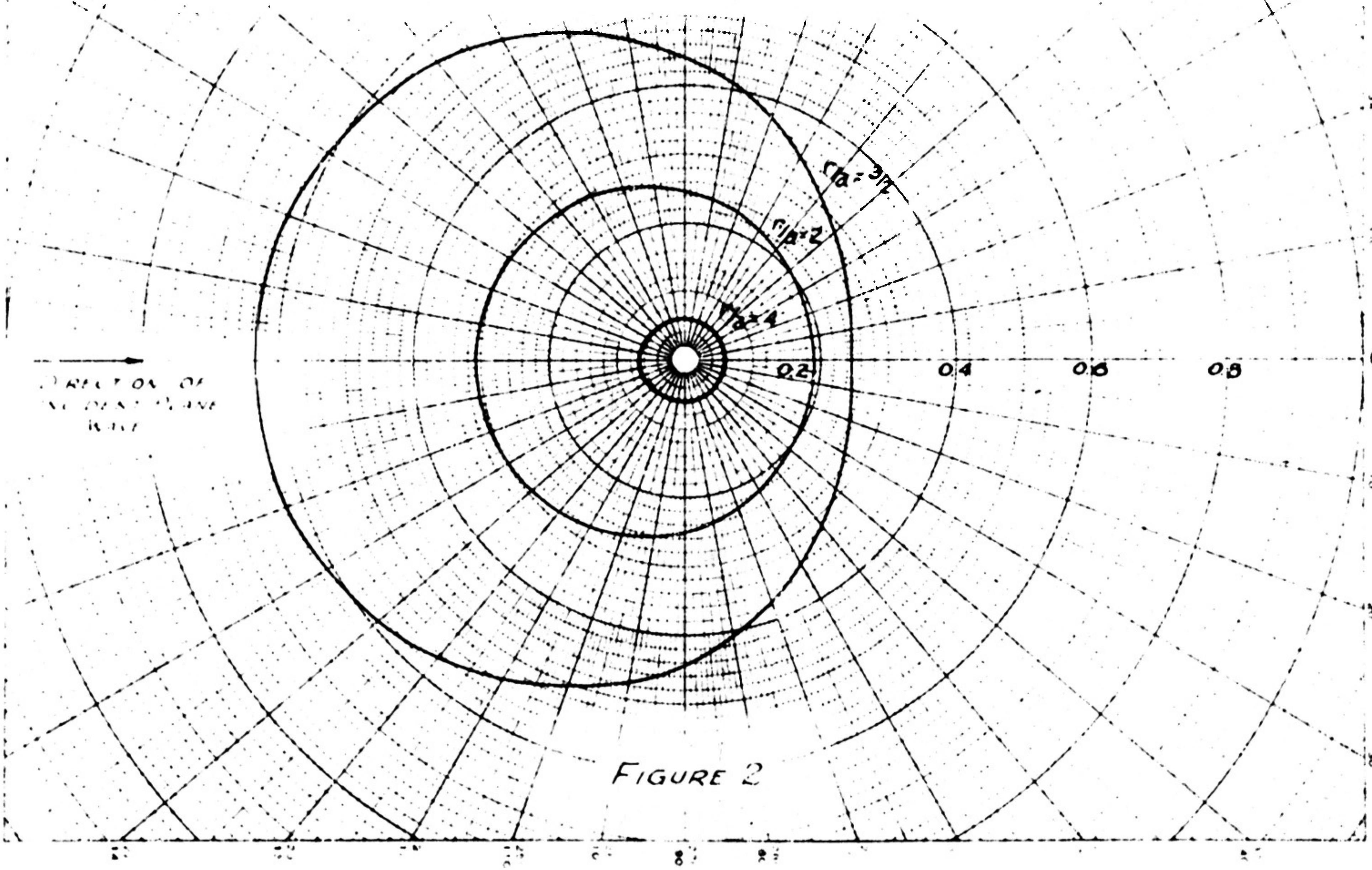
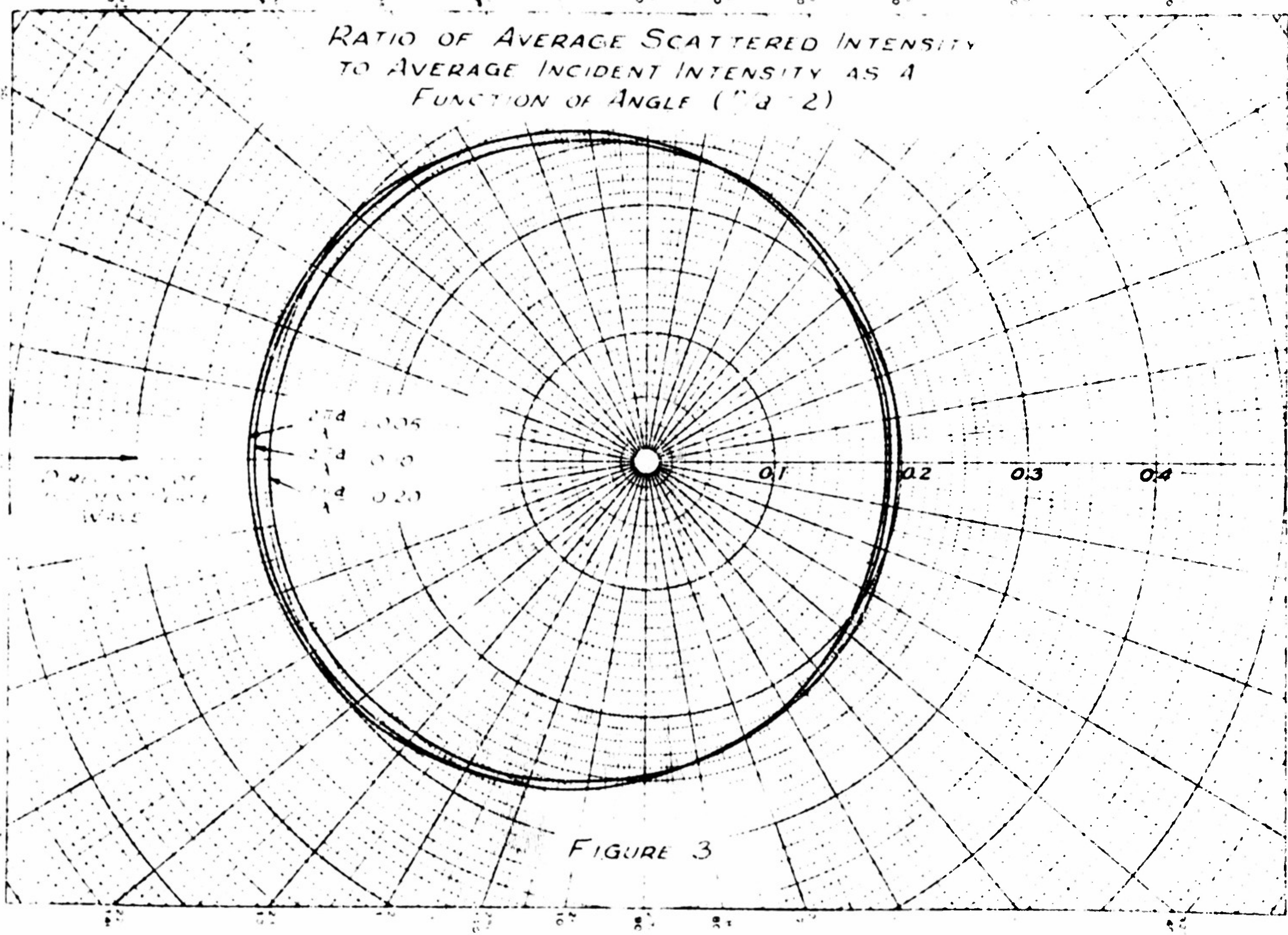


FIGURE 2



RATIO OF AVERAGE SCATTERED INTENSITY TO AVERAGE INCIDENT INTENSITY AS A FUNCTION OF ANGLE

FIGURE 3

INTENSITY DEVIATION ANGLE Δ
AS A FUNCTION OF POLAR ANGLE θ
$ka = 0.1$ $r/a = 2$

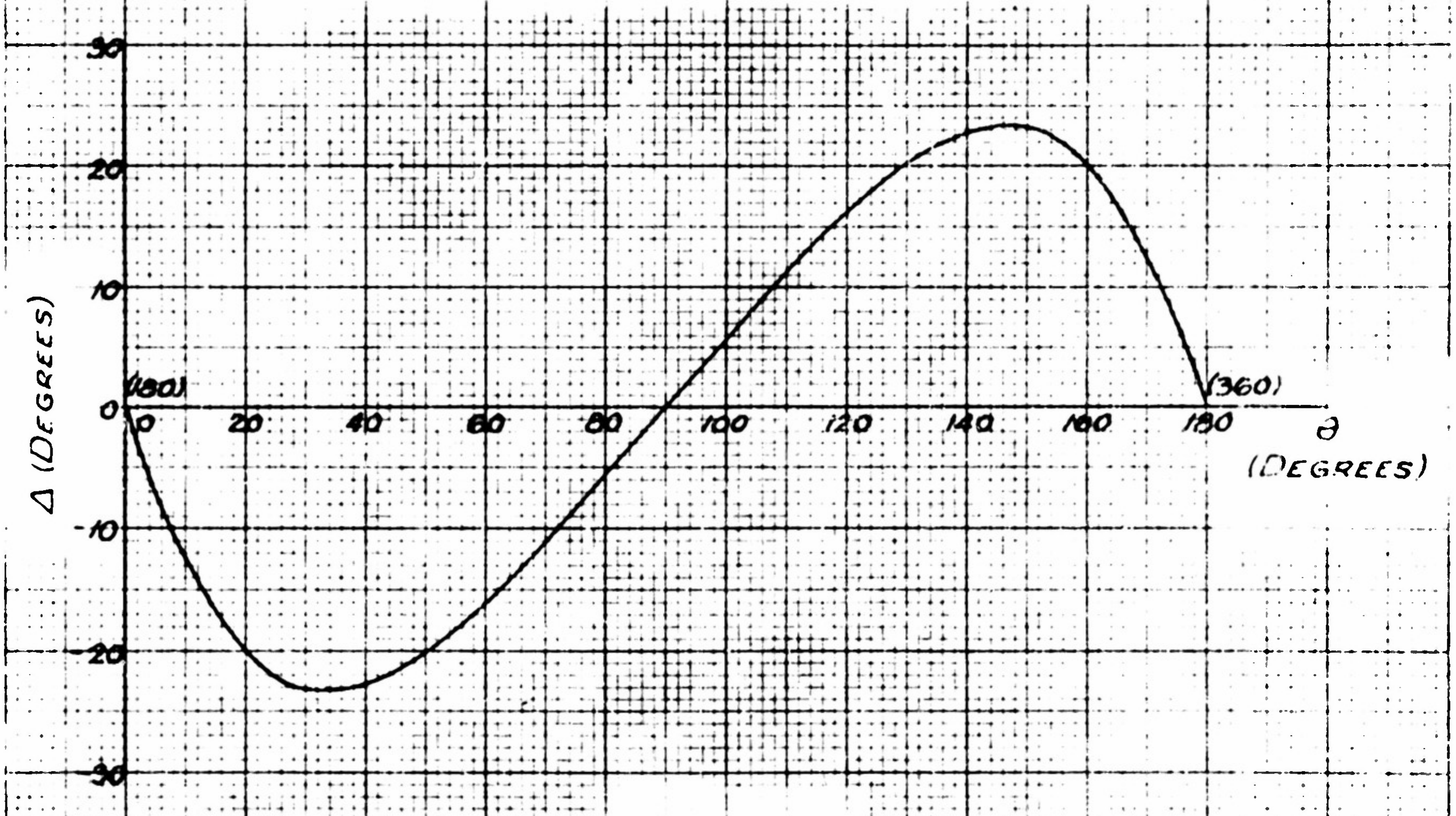


FIGURE 4

RATIO OF AVERAGE TOTAL INTENSITY TO AVERAGE INCIDENT INTENSITY AS A FUNCTION OF ANGLE

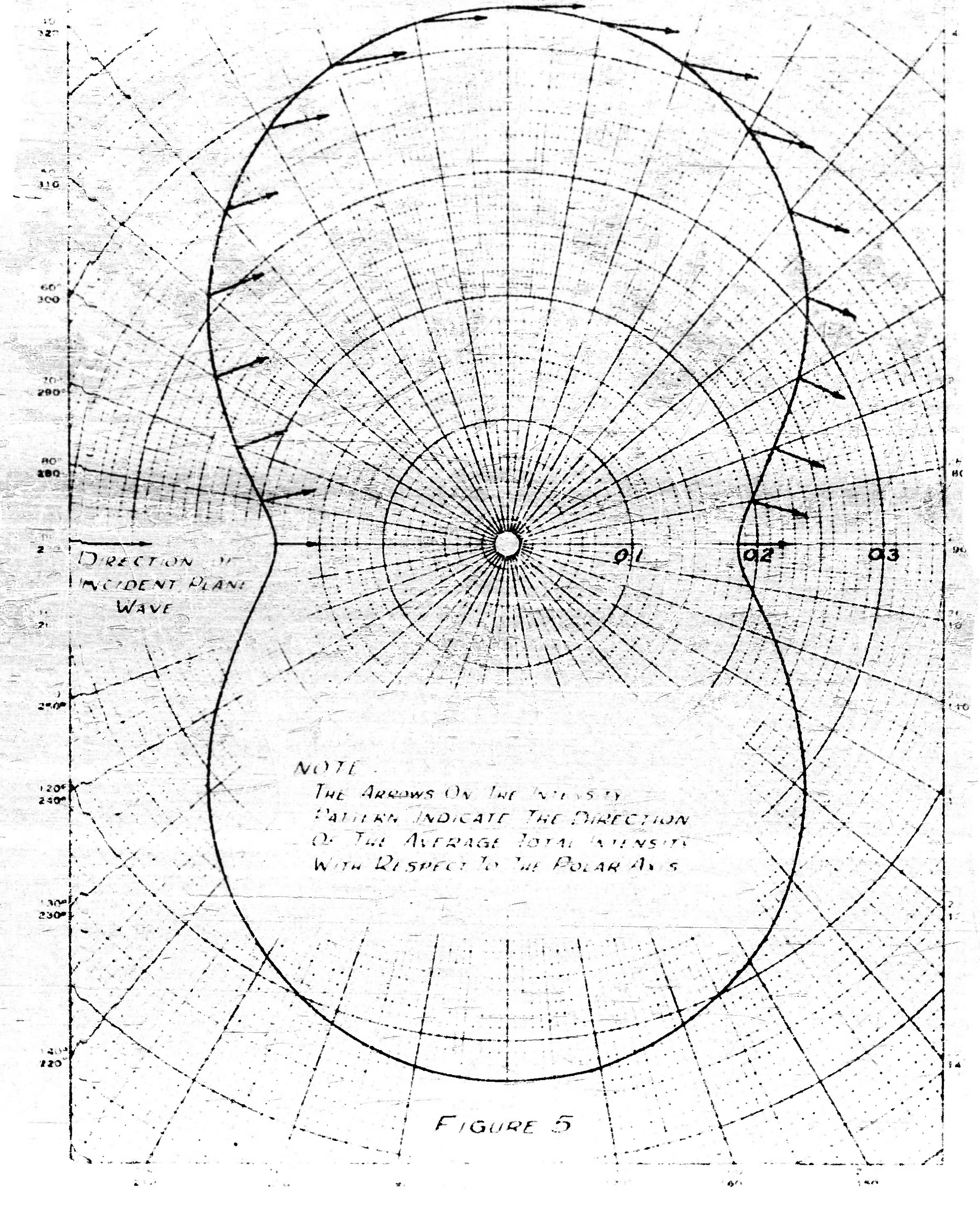


FIGURE 5

RATIO OF AVERAGE TOTAL INTENSITY TO AVERAGE INCIDENT INTENSITY AS A FUNCTION OF ANGLE FOR ka = 0 AND r/a = 2, 4, 10, ∞

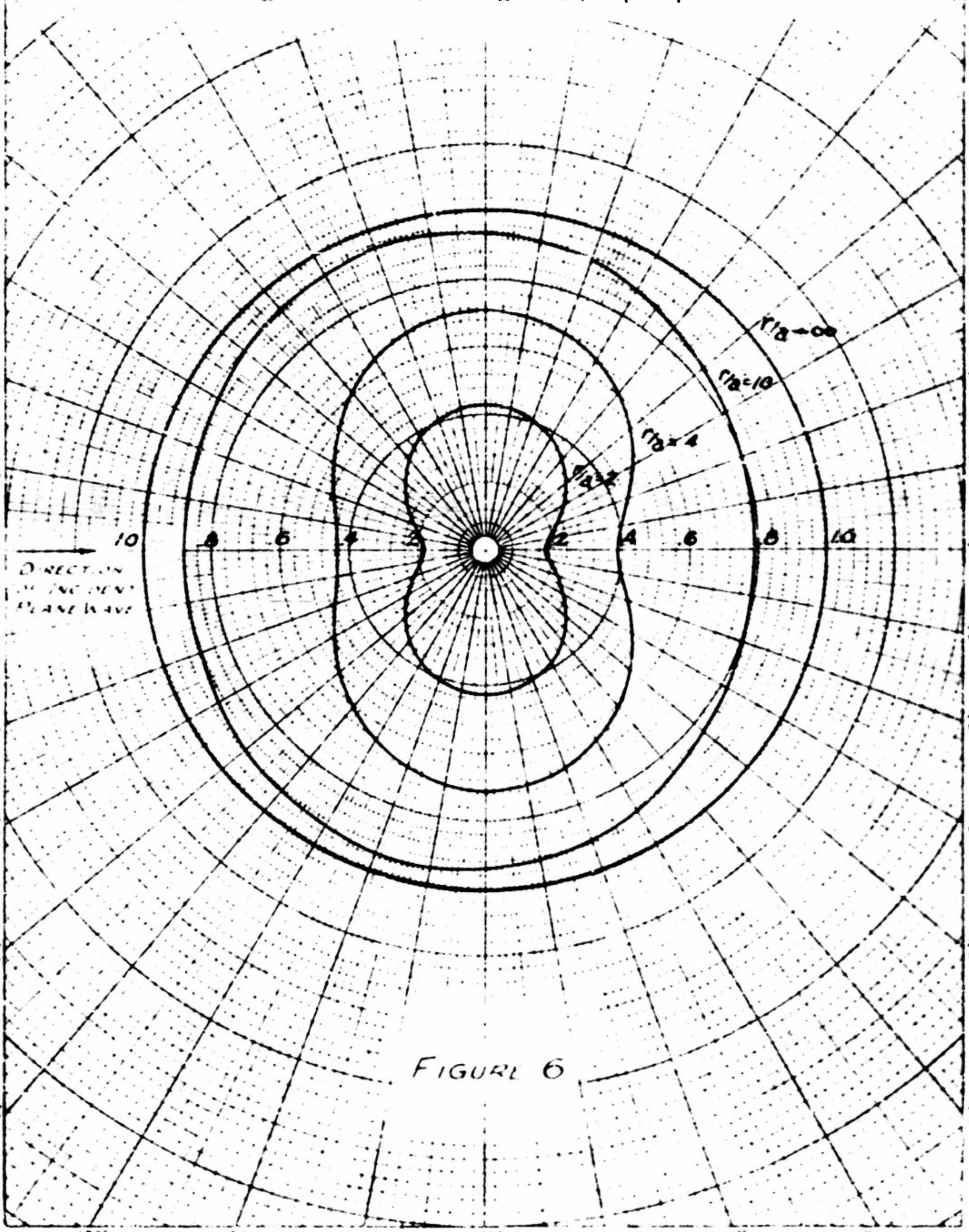


FIGURE 6